

Signals and Systems

Properties of the Fourier transform

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Properties of the Fourier transform – linearity

- All four transforms are linear:

$$af(t) + bg(t) \xleftrightarrow{CTFT} aF(j\omega) + bG(j\omega)$$

$$a\tilde{f}(t) + b\tilde{g}(t) \xleftrightarrow{CTFS; \omega_0} aF_k + bG_k$$

$$af[n] + bg[n] \xleftrightarrow{DTFT} aF(e^{j\Omega}) + bG(e^{j\Omega})$$

$$a\tilde{f}[n] + b\tilde{g}[n] \xleftrightarrow{DTFS; \Omega_0} aF_k + bG_k$$

The periodic signals $\tilde{f}$ i $\tilde{g}$ have the same period $T_0 = \frac{2\pi}{\omega_0}$ or $N = \frac{2\pi}{\Omega_0}$, respectively.

- Proof for the *DTFT*:

$$\begin{aligned} DTFT\{af[n] + bg[n]\} &= \sum_{n=-\infty}^{\infty} \{af[n] + bg[n]\}e^{-j\Omega n} \\ &= a \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} + b \sum_{n=-\infty}^{\infty} g[n]e^{-j\Omega n} = aF(e^{j\Omega}) + bG(e^{j\Omega}) \end{aligned}$$

Properties of the Fourier transform – symmetry

- Let us analyze symmetry of the *CTFT*
- From $F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$, we have:

$$F^*(j\omega) = \left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \right]^* = \int_{-\infty}^{\infty} f^*(t)e^{j\omega t} dt. \quad (1)$$

For a real signal, $f(t) = f^*(t)$, $\forall t \in \mathbb{R}$. Hence:

$$F^*(j\omega) = \underbrace{\int_{-\infty}^{\infty} f(t)e^{-j(-\omega)t} dt}_{F(-j\omega)},$$

which leads to conjugate symmetry of the complex spectrum $F(j\omega)$:

$$F^*(j\omega) = F(-j\omega)$$

Properties of the Fourier transform – symmetry

- Conjugate symmetry corresponds to:

$$\operatorname{Re}\{F(j\omega)\} = \operatorname{Re}\{F(-j\omega)\} \quad \text{and} \quad \operatorname{Im}\{F(j\omega)\} = -\operatorname{Im}\{F(-j\omega)\},$$

as well as:

$$|F(j\omega)| = |F(-j\omega)| \quad \text{and} \quad \angle F(j\omega) = -\angle F(-j\omega).$$

- The amplitude spectrum of a real signal is even, and the phase spectrum of a real signal is an odd function. The real part of the spectrum of a real signal is even, and the imaginary part is odd.
- All Fourier transforms of real signals exhibit conjugate symmetry of its spectra.

Properties of the Fourier transform – symmetry

- Let us analyze the spectrum of a real and even continuous signal. Since $f^*(t) = f(t)$ and $f(t) = f(-t)$, we have:

$$f^*(t) = f(-t).$$

From equation (1), it follows:

$$\begin{aligned} F^*(j\omega) &= \int_{-\infty}^{\infty} f^*(t) e^{j\omega t} dt = \int_{-\infty}^{\infty} f(-t) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} f(-t) e^{-j(-\omega)t} dt \stackrel{|\tau = -t|}{=} \\ &= \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau = F(j\omega). \end{aligned}$$

- Such an equality stands only if the imaginary part of $F(j\omega)$ is zero, which means that $F(j\omega)$ is real. Hence, spectrum of an **even** real signal is **real**.

Properties of the Fourier transform – symmetry

- Symmetry properties of all Fourier transforms

transform	real signal	even real signal
<i>CTFT</i>	$F^*(j\omega) = F(-j\omega)$	$F^*(j\omega) = F(j\omega)$
<i>CTFS</i>	$F_k^* = F_{-k}$	$F_k^* = F_k$
<i>DTFT</i>	$F^*(e^{j\Omega}) = F(e^{-j\Omega})$	$F^*(e^{j\Omega}) = F(e^{j\Omega})$
<i>DTFS</i>	$F_k^* = F_{-k}$	$F_k^* = F_k$
	spectrum is conjugate symmetric	spectrum is real

Convolution of discrete-time signals

- Convolution is a very important operation on signals, which will be frequently exploited for calculation of the response of linear systems.
- Let $x, y \in [\mathbb{Z} \rightarrow \mathbb{R}]$ be two discrete-time signals.
- The result of the convolution of x and y is the discrete-time signal $x * y$, defined by the convolution sum:

$$\forall n \in \mathbb{Z}, \quad (x * y)[n] = \sum_{m=-\infty}^{\infty} x[m]y[n - m].$$

Convolution of discrete-time signals – properties

- Convolution is commutative:

$$(x * y)[n] = (y * x)[n],$$

- and linear. For $a_1, a_2 \in \mathbb{R}$ we have:

$$x * (a_1 y_1 + a_2 y_2) = a_1 (x * y_1) + a_2 (x * y_2).$$

Properties of the Fourier transform – convolution of aperiodic signals in the time domain

- We calculate the convolution of two aperiodic discrete-time signals $f[n]$ and $g[n]$, $\forall n \in \mathbb{Z}$, whose spectra are $F(e^{j\Omega})$ and $G(e^{j\Omega})$, $\forall \Omega \in \mathbb{R}$

$$\begin{aligned} DTFT\{(f * g)[n]\} &= DTFT\left\{\sum_{m=-\infty}^{\infty} f[m]g[n-m]\right\} = \\ &= F(e^{j\Omega})G(e^{j\Omega}). \end{aligned}$$

Proof:

$$\begin{aligned} DTFT\{(f * g)[n]\} &= \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} f[m]g[n-m] \right] e^{-j\Omega n} = \\ &= \sum_{m=-\infty}^{\infty} f[m] \left[\sum_{n=-\infty}^{\infty} g[n-m] e^{-j\Omega n} \right] \end{aligned}$$

substitution $n - m = k$ gives ...

Properties of the Fourier transform – convolution of aperiodic signals in the time domain

$$\begin{aligned}DTFT\{(f * g)[n]\} &= \sum_{m=-\infty}^{\infty} f[m] \left[\sum_{k=-\infty}^{\infty} g[k] e^{-j\Omega(m+k)} \right] \\&= \underbrace{\sum_{m=-\infty}^{\infty} f[m] e^{-j\Omega m}}_{F(e^{j\Omega})} \left[\underbrace{\sum_{k=-\infty}^{\infty} g[k] e^{-j\Omega k}}_{G(e^{j\Omega})} \right] \\&= F(e^{j\Omega}) G(e^{j\Omega})\end{aligned}$$

- Similarly, we show that convolution of aperiodic continuous-time signals corresponds to multiplication of its spectra:

$$CTFT\{(f * g)(t)\} = F(j\omega)G(j\omega)$$

Convolution of aperiodic signals in the time domain

- Let us calculate the spectrum of two convolved rectangular pulses.

$$p_L[n] = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{for other } n \end{cases},$$

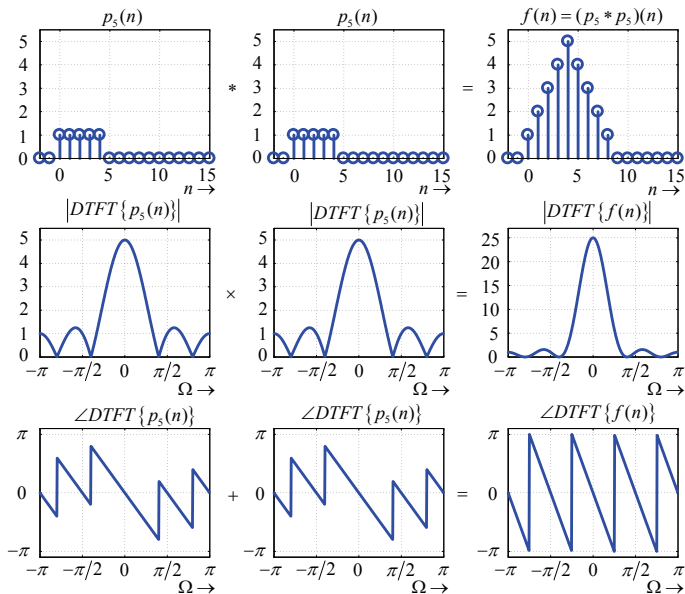
$$DTFT\{p_L[n]\} = \begin{cases} L, & \Omega = 0, \pm 2\pi, \pm 4\pi, \dots \\ e^{-j\frac{\Omega}{2}(L-1)} \frac{\sin(\frac{\Omega L}{2})}{\sin(\frac{\Omega}{2})}, & \Omega \neq 0, \pm 2\pi, \pm 4\pi, \dots \end{cases}$$

$$DTFT\{(p_L * p_L)[n]\} = DTFT\{p_L[n]\} DTFT\{p_L[n]\}$$

$$DTFT\{(p_L * p_L)[n]\} = \begin{cases} L^2, & \Omega = 0, \pm 2\pi, \dots \\ \left[e^{-j\frac{\Omega}{2}(L-1)} \frac{\sin(\frac{\Omega L}{2})}{\sin(\frac{\Omega}{2})} \right]^2, & \Omega \neq 0, \pm 2\pi, \dots \end{cases}$$

$$DTFT\{(p_5 * p_5)[n]\} = \begin{cases} 25, & \Omega = 0, \pm 2\pi, \pm 4\pi, \dots \\ e^{-j4\Omega} \frac{\sin^2(\frac{5\Omega}{2})}{\sin^2(\frac{\Omega}{2})}, & \Omega \neq 0, \pm 2\pi, \pm 4\pi, \dots \end{cases}$$

Convolution of aperiodic signals in the time domain



Convolution of continuous-time signals

- Let $x, y \in [\mathbb{R} \rightarrow \mathbb{R}]$ be two continuous-time signals.
- Convolution $conv$ is a mapping of x, y to a new z signal:

$$conv : [\mathbb{R} \rightarrow \mathbb{R}] \times [\mathbb{R} \rightarrow \mathbb{R}] \rightarrow [\mathbb{R} \rightarrow \mathbb{R}]$$
$$z = conv(x, y) = x * y.$$

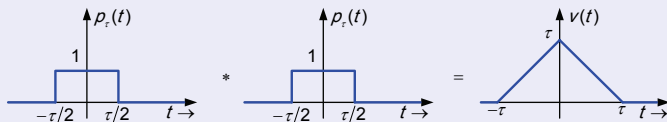
- It is defined by the convolution integral:

$$\forall t \in \mathbb{R}, \quad z(t) = (x * y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t - \tau) d\tau.$$

- Continuous convolution is commutative and linear, as well.

Convolution of continuous-time signals – example

- The result of the convolution^a $v = p_\tau * p_\tau$ is:



^aStudents are encouraged to solve it using the convolution integral from the previous slide.

Properties of the Fourier transform of periodic signals

- For the periodic discrete-time signals f and g with the same period N , we define the periodic convolution:

$$(f \circledast g)[n] = \sum_{m=0}^{N-1} f[m]g[n-m]$$

- For the periodic continuous-time signals f and g with the same period T_0 , we define the periodic convolution:

$$(f \circledast g)(t) = \int_0^{T_0} f(\tau)g(t-\tau) d\tau$$

- It can be shown that:

$$\begin{aligned}(f \circledast g)[n] &\xleftrightarrow{DTFS; \Omega_0 = \frac{2\pi}{N}} NF_k G_k \\(f \circledast g)(t) &\xleftrightarrow{CTFS; \omega_0 = \frac{2\pi}{T_0}} T_0 F_k G_k\end{aligned}$$

Properties of the Fourier transform – convolution in the time domain

- Convolution properties of all Fourier transforms

$(f * g)(t)$	$\xleftrightarrow{CTFT}$	$F(j\omega)G(j\omega)$
$(f \circledast g)(t)$	$\xleftrightarrow{CTFS; \omega_0 = \frac{2\pi}{T_0}}$	$T_0 F_k G_k$
$(f * g)[n]$	$\xleftrightarrow{DTFT}$	$F(e^{j\Omega})G(e^{j\Omega})$
$(f \circledast g)[n]$	$\xleftrightarrow{DTFS; \Omega_0 = \frac{2\pi}{N}}$	$N F_k G_k$

Shift in the time domain – discrete signals

- For $F(e^{j\Omega}) = DTFT\{f[n]\}$, $\forall n \in \mathbb{Z}$, $\forall \Omega \in \mathbb{R}$, we have:

$$DTFT\{f[n - m]\} = e^{-j\Omega m} F(e^{j\Omega}).$$

Proof:

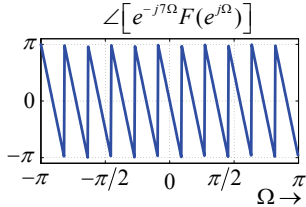
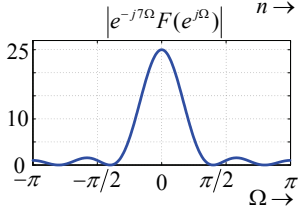
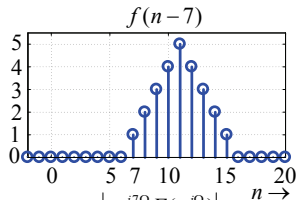
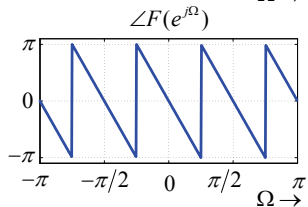
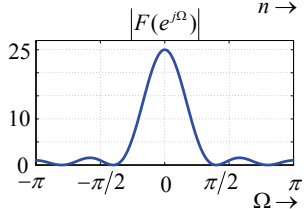
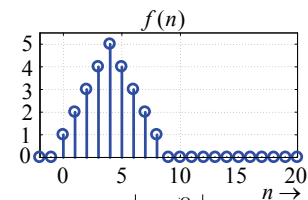
$$DTFT\{f[n - m]\} = \sum_{n=-\infty}^{\infty} f[n - m] e^{-j\Omega n}$$

substitution $n - m = r$ gives

$$= \sum_{r=-\infty}^{\infty} f[r] e^{-j\Omega(m+r)}$$

$$= e^{-j\Omega m} \underbrace{\sum_{r=-\infty}^{\infty} f[r] e^{-j\Omega r}}_{F(e^{j\Omega})} = e^{-j\Omega m} F(e^{j\Omega})$$

Discrete aperiodic signal – shift in the time domain



Properties – shift in the time domain

- Time-shift properties of all Fourier transforms

$f(t - t_1)$	$\xleftrightarrow{CTFT}$	$e^{-j\omega t_1} F(j\omega)$
$f(t - t_1)$	$\xleftrightarrow{CTFS; \omega_0}$	$e^{-jk\omega_0 t_1} F_k$
$f[n - m]$	$\xleftrightarrow{DTFT}$	$e^{-j\Omega m} F(e^{j\Omega})$
$f[n - m]$	$\xleftrightarrow{DTFS; \Omega_0}$	$e^{-jk\Omega_0 m} F_k$

Properties – shift in the frequency domain

- For $F(e^{j\Omega}) = DTFT\{f[n]\}$, $\forall n \in \mathbb{Z}$, $\forall \Omega \in \mathbb{R}$, we have:

$$DTFT\{e^{j\Omega_1 n} f[n]\} = F(e^{j(\Omega - \Omega_1)}).$$

Proof:

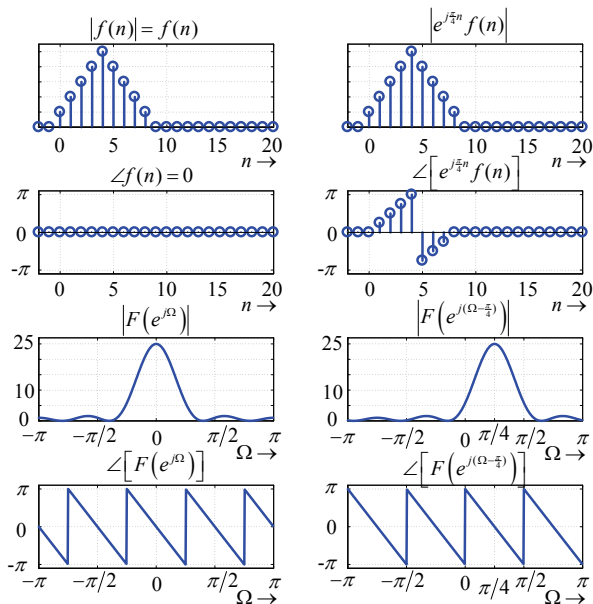
$$DTFT\{e^{j\Omega_1 n} f[n]\} = \sum_{n=-\infty}^{\infty} e^{j\Omega_1 n} f[n] e^{-j\Omega n}$$

$$DTFT\{e^{j\Omega_1 n} f[n]\} = \sum_{n=-\infty}^{\infty} f[n] e^{-j(\Omega - \Omega_1)n} = F(e^{j(\Omega - \Omega_1)})$$

Example: $\Omega_1 = \frac{\pi}{4}$

$$DTFT\{e^{j\frac{\pi}{4} n} f[n]\} = \sum_{n=-\infty}^{\infty} f[n] e^{-j(\Omega - \frac{\pi}{4})n} = F(e^{j(\Omega - \frac{\pi}{4})})$$

Discrete signal – shift in the frequency domain



Properties – shift in the frequency domain

- Frequency-shift properties of all Fourier transforms

$e^{j\omega_1 t} f(t)$	$\xleftrightarrow{CTFT}$	$F(j(\omega - \omega_1))$
$e^{jk_1\omega_0 t} f(t)$	$\xleftrightarrow{CTFS; \omega_0}$	F_{k-k_1}
$e^{j\Omega_1 n} f[n]$	$\xleftrightarrow{DTFT}$	$F(e^{j(\Omega - \Omega_1)})$
$e^{jk_1\Omega_0 n} f[n]$	$\xleftrightarrow{DTFS; \Omega_0}$	F_{k-k_1}

Periodic convolution in the frequency domain

- If $F(e^{j\Omega}) = DTFT\{f[n]\}$, and $G(e^{j\Omega}) = DTFT\{g[n]\}$, $\forall n \in \mathbb{Z}$, $\forall \Omega \in \mathbb{R}$, then:

$$f[n]g[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\psi}) G(e^{j(\Omega-\psi)}) d\psi = \frac{1}{2\pi} (F \circledast G)(e^{j\Omega})$$

Proof:

$$\begin{aligned} DTFT\{f[n]g[n]\} &= \sum_{n=-\infty}^{\infty} \{f[n]g[n]\} e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\psi}) e^{j\psi n} d\psi \right] g[n] e^{-j\Omega n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\psi}) \left[\sum_{n=-\infty}^{\infty} g[n] e^{-j(\Omega-\psi)n} \right] d\psi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\psi}) G(e^{j(\Omega-\psi)}) d\psi = \frac{1}{2\pi} (F \circledast G)(e^{j\Omega}) \end{aligned}$$

Convolution in the frequency domain

- Using $F(j\omega) = CTFT\{f(t)\}$, and $G(j\omega) = CTFT\{g(t)\}$, $\forall t \in \mathbb{R}$, $\forall \omega \in \mathbb{R}$, we have

$$f(t)g(t) \xleftrightarrow{CTFT} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j(\omega - \Psi))G(j\Psi) d\Psi = \frac{1}{2\pi} (F * G)(j\omega)$$

- The proof is similar to the previous one.

Fourier transform of the complex exponential

- We start from some signal $f(t)$, $\forall t \in \mathbb{R}$. Let its Fourier transform be a Dirac function with an intensity of 2π at the frequency ω_0 .
- The *ICTFT* of such an impulse is:

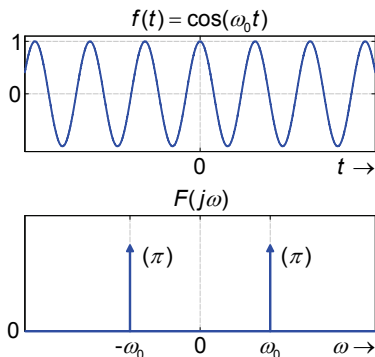
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

- Clearly, such a signal is the complex exponential.

Fourier transform of the cosine

- Using Euler's formula for $f(t) = \cos(\omega_0 t)$, $\forall t \in \mathbb{R}$, and using the result from previous slide, we have:

$$\begin{aligned}CTFT\{\cos(\omega_0 t)\} &= CTFT\left\{\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right\} = \\&= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)\end{aligned}$$



Convolution in the frequency domain

- We consider a product of infinite duration sinusoidal signal $f(t) = \cos(\omega_0 t)$, $\forall t \in \mathbb{R}$, and some aperiodic continuous-time signal $g(t)$, $\forall t \in \mathbb{R}$, with the spectrum $G(j\omega)$, $\forall \omega \in \mathbb{R}$.
- Since

$$F(j\omega) = CTFT\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

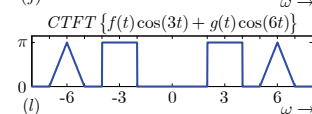
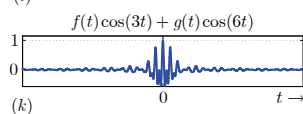
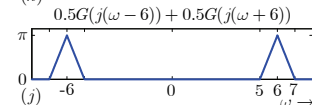
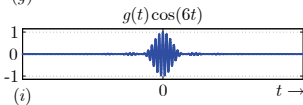
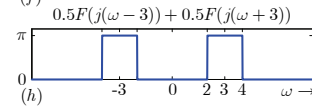
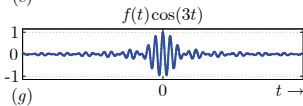
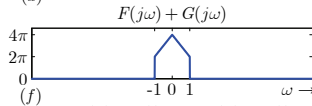
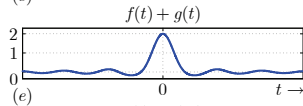
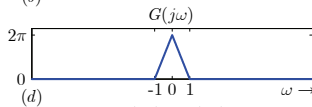
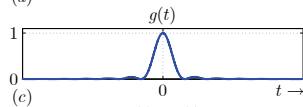
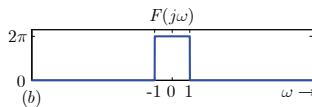
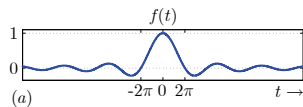
we have:

$$\begin{aligned} CTFT\{\cos(\omega_0 t)g(t)\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\psi)G(j(\omega - \psi)) d\psi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\pi\delta(\psi + \omega_0) + \pi\delta(\psi - \omega_0)]G(j(\omega - \psi)) d\psi \\ &= \frac{1}{2}G(j(\omega + \omega_0)) + \frac{1}{2}G(j(\omega - \omega_0)) \end{aligned}$$

Amplitude modulation

- Calculating the product $g(t) \cos(\omega_0 t)$ is called amplitude modulation, where $\cos(\omega_0 t)$ is the carrier, and $g(t)$ is a signal that contains some information.
- Amplitude modulation shifts the spectrum of the signal, thus enabling simultaneous transfer of information from different sources over the same communication channel.

Amplitude modulation – illustration



Properties – Convolution in the frequency domain

- Frequency domain convolution of all Fourier transforms

$f(t)g(t)$	$\xleftrightarrow{CTFT}$	$\frac{1}{2\pi}(F * G)(j\omega)$
$f(t)g(t)$	$\xleftrightarrow{CTFS; \omega_0=2\pi/T_0}$	$(F * G)_k$
$f[n]g[n]$	$\xleftrightarrow{DTFT}$	$\frac{1}{2\pi}(F \circledast G)(e^{j\Omega})$
$f[n]g[n]$	$\xleftrightarrow{DTFS; \Omega_0=2\pi/N}$	$(F \circledast G)_k$

Properties of the Fourier transform – duality

- Showing that $f(t) \xleftrightarrow{CTFT} F(j\omega)$ has the property of duality

$$F(jt) \xleftrightarrow{CTFT} 2\pi f(-\omega) \quad \text{or} \quad F(-jt) \xleftrightarrow{CTFT} 2\pi f(\omega)$$

- Proof is based on the similarity of the *CTFT* transform pair equations, with successive substitutions of variables:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

By substituting $t = -\tau$, and multiplying both sides with 2π , it follows:

$$2\pi f(-\tau) = \int_{-\infty}^{\infty} F(j\omega) e^{-j\omega\tau} d\omega$$

Properties of the Fourier transform – duality

Substituting $\omega = t$ gives

$$2\pi f(-\tau) = \int_{-\infty}^{\infty} F(jt)e^{-jt\tau} dt$$

Finally, the substitution $\tau = \omega$ leads to:

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)e^{-j\omega t} dt = CTFT\{F(jt)\}$$

which proves

$$F(jt) \xleftrightarrow{CTFT} 2\pi f(-\omega)$$

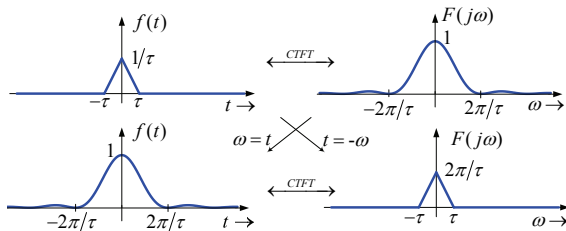
Similarly, $F(-jt) \xleftrightarrow{CTFT} 2\pi f(\omega)$

Properties – duality

- Duality of all Fourier transforms

<i>CTFT</i>	$f(t) \xleftrightarrow{CTFT} F(j\omega)$	$F(jt) \xleftrightarrow{CTFT} 2\pi f(-\omega)$
<i>DTFS</i>	$f[n] \xleftrightarrow{DTFS; 2\pi/N} F_k$	$F_n \xleftrightarrow{DTFS; \Omega_0=2\pi/N} \frac{1}{N} f[-k]$
<i>DTFT</i>	$f[n] \xleftrightarrow{DTFT} F(e^{j\Omega})$	$F(e^{jt}) \xleftrightarrow{CTFS; \omega_0=1} f[-k]$
<i>CTFS</i>		

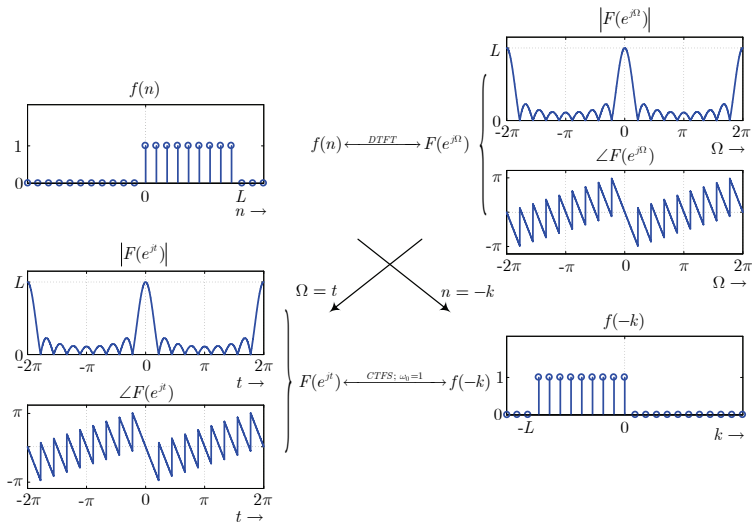
CTFT – duality – example



$$\underbrace{\left. \begin{array}{ll} \frac{1}{\tau^2}t + \frac{1}{\tau} & -\tau \leq t < 0 \\ -\frac{1}{\tau^2}t + \frac{1}{\tau} & 0 \leq t \leq \tau \\ 0 & \text{elsewhere} \end{array} \right\}}_{f(t)} \xleftrightarrow{\text{CTFT}} \underbrace{\left[\frac{\sin \frac{\tau\omega}{2}}{\frac{\tau\omega}{2}} \right]^2}_{F(j\omega)}$$

$$\underbrace{\left[\frac{\sin \frac{\tau t}{2}}{\frac{\tau t}{2}} \right]^2}_{F(jt)} \xleftrightarrow{\text{CTFT}} \underbrace{\left\{ \begin{array}{ll} \frac{2\pi}{\tau^2}\omega + \frac{2\pi}{\tau} & -\tau \leq \omega < 0 \\ -\frac{2\pi}{\tau^2}\omega + \frac{2\pi}{\tau} & 0 \leq \omega \leq \tau \\ 0 & \text{elsewhere} \end{array} \right\}}_{2\pi f(-\omega) = 2\pi f(\omega)}$$

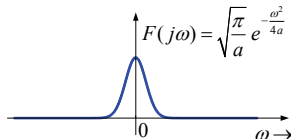
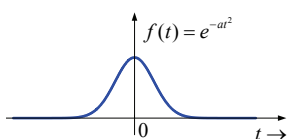
DTFT/CTFS – duality – example



Fourier transform – Gaussian

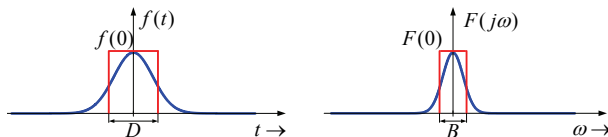
- The Fourier transform of an aperiodic continuous-time **Gaussian** signal $f(t) = e^{-at^2}$, $\forall t \in \mathbb{R}$ is **Gaussian**, as well:

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-(at^2 + j\omega t)} dt = \\ &= \int_{-\infty}^{\infty} e^{-[(\sqrt{a}t + \frac{j\omega}{2\sqrt{a}})^2 + \frac{\omega^2}{4a}]} dt = e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-(\sqrt{a}t + \frac{j\omega}{2\sqrt{a}})^2} dt \\ &= \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{4a}} \underbrace{\int_{-\infty}^{\infty} e^{-\beta^2} d\beta}_{\sqrt{\pi}} = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \end{aligned}$$



Signal duration and the spectrum width

- There are many possible ways to define the duration and spectrum width of some signal. We use Equal-Area Rectangle definition.
- Our example is a real and even signal:



- The duration of signal $f(t)$ is, by definition, the duration of the rectangular pulse of the same height and area as the $f(t)$.
- The same approach is used for the spectrum width.

Signal duration and the spectrum width

- Using

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \quad \& \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t} d\omega$$

and

$$\underbrace{\int_{-\infty}^{\infty} f(t) dt}_{F(0)} = Df(0) \quad \& \quad \underbrace{\int_{-\infty}^{\infty} F(j\omega) d\omega}_{2\pi f(0)} = BF(0)$$

it follows

$$D = \frac{F(0)}{f(0)} \quad \& \quad B = 2\pi \frac{f(0)}{F(0)} \quad \Rightarrow \quad DB = 2\pi$$

Signal duration and the spectrum width

- The duration of the signal and width of its spectrum are reciprocal
- A shorter signal means wider spectrum, and vice versa
- The product:
 $(\text{signal duration}) \times (\text{spectrum width}) = \text{constant}$
- This is a very important fact in many applications, especially in communications.

Contents

1 Properties of the Fourier transform

- Linearity
- Symmetry
- Convolution in the time domain
- Shift in the time domain
- Shift in the frequency domain
- Convolution in the frequency domain
- Duality
- Spectrum of finite and infinite signals

2 Appendix

- Transform tables, properties – CTFT i DTFT

Basic transforms (CTFT)

Time domain: $x(t)$	Freq. domain: $X(j\omega)$
$\delta(t)$	1
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\mu(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\mu(t + \frac{\tau}{2}) - \mu(t - \frac{\tau}{2})$	$\tau \frac{\sin \frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}$
$\omega_1 \frac{\sin \frac{\omega_1 t}{2}}{\frac{\omega_1 t}{2}}$	$2\pi [\mu(\omega + \frac{\omega_1}{2}) - \mu(\omega - \frac{\omega_1}{2})]$
$e^{-bt}\mu(t), \quad b > 0$	$\frac{1}{b + j\omega}$
$A \cos(\omega_0 t + \phi)$	$\pi A e^{j\phi} \delta(\omega - \omega_0) + \pi A e^{-j\phi} \delta(\omega + \omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin(\omega_0 t)$	$-j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$
$\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$	$\sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\omega - k\omega_0)$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$

Properties (CTFT)

Property	Time domain: $x(t)$	Freq. domain: $X(j\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(j\omega) + bX_2(j\omega)$
Conjugate	$x^*(t)$	$X^*(-j\omega)$
T. reversal	$x(-t)$	$X(-j\omega)$
Duality	$X(jt)$	$2\pi x(-\omega)$
Duality	$X(-jt)$	$2\pi x(\omega)$
Time scale	$x(at)$	$\frac{1}{ a } X(\frac{j\omega}{a})$
Time shift	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
C. modulation	$x(t)e^{j\omega t_0}$	$X(j(\omega - \omega_0))$
Modulation	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$
Derivative	$\frac{d^k x(t)}{dt^k}$	$(j\omega)^k X(j\omega)$
Convolution	$(x_1 * x_2)(t)$	$X_1(j\omega)X_2(j\omega)$
Product	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$

Basic transforms (DTFT)

Time domain: $x[n]$	Freq. domain: $X(e^{j\Omega})$
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\Omega n_0}$
$\mu[n]$	$\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m) + \frac{1}{1 - e^{-j\Omega}}$
1	$2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$
$e^{j\omega_0 n}$	$2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$
$\begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{for other } n \end{cases}$	$e^{-j\frac{\Omega}{2}(L-1)} \frac{\sin(\frac{\Omega L}{2})}{\sin(\frac{\Omega}{2})}$
$\alpha^n \mu[n], \quad \alpha < 1$	$\frac{1}{1 - \alpha e^{-j\Omega}}$
$(n+1)\alpha^n \mu[n], \quad \alpha < 1$	$\frac{1}{(1 - \alpha e^{-j\Omega})^2}$
$\cos[\Omega_0 n]$	$\pi \sum_{m=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi m) + \pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$
$\sin[\Omega_0 n]$	$j\pi \sum_{m=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi m) - j\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$

Properties (DTFT)

Property	Time domain: $x[n]$	Freq. domain: $X(e^{j\Omega})$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\Omega}) + bX_2(e^{j\Omega})$
Conjugate	$x^*[n]$	$X^*(e^{-j\Omega})$
Time reversal	$x[-n]$	$X(e^{-j\Omega})$
Time shift	$x[n - n_0]$	$e^{-j\Omega n_0} X(e^{j\Omega})$
Freq. shift	$x[n]e^{j\Omega_0 n}$	$X(e^{j(\Omega - \Omega_0)})$
Modulation	$x[n] \cos[\Omega_0 n]$	$\frac{1}{2}X(e^{j(\Omega + \Omega_0)}) + \frac{1}{2}X(e^{j(\Omega - \Omega_0)})$
Derivative in freq.	$nx[n]$	$(j \frac{dX(e^{j\Omega})}{d\Omega})$
Convolution	$(x_1 * x_2)[n]$	$X_1(e^{j\Omega})X_2(e^{j\Omega})$
Product	$x_1[n]x_2[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\psi})X_2(e^{j(\Omega - \psi)}) d\psi$
Conjugate symmetry of real signals	$x[n]$ real signal	$X(e^{j\Omega}) = X^*(e^{-j\Omega})$ $\text{Re}\{X(e^{j\Omega})\} = \text{Re}\{X(e^{-j\Omega})\}$ $\text{Im}\{X(e^{j\Omega})\} = -\text{Im}\{X(e^{-j\Omega})\}$ $ X(e^{j\Omega}) = X(e^{-j\Omega}) $ $\angle X(e^{j\Omega}) = -\angle X(e^{-j\Omega})$
Symmetry of real and even signals	$x[n]$ real and even	$X(e^{j\Omega})$ real and even